ساختارهای طبیعی روی منیفلدها

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چکیده

بسیاری از ساختارها روی منیفلدها را ممکن‌است به‌وسیلهٔ اطلسی از دستگاه مختصات موضعی تعریف کرد به‌قسمی‌که تبدیلات مختصاتی چارتهای متقاطع دارای خاصیت ویژه‌ای باشند، مثلا این تبدیلات تشکیل یک شبه‌گروه روی R^n بدهند.

در این مقاله، هدف ما معرفی ساختارهای جدیدی روی منیفلدها است بنام «ساختار طبیعی» که‌به وسیله‌اطلسی از دستگاه مختصات طبیعی تشکیل می‌شود.

مکانیسم دستگاه مختصات در این ساختارها به‌قسمی است‌که ساختار خطی R^n را به نحو مناسبی به‌منیفلدها منتقل می‌کند، مثلا زیرفضاهای یک بعدی‌را روی ژئودزیک‌های منیفلد تصویر می‌کند و بدین ترتیب رابطه‌ای مستقیم بین معادلهٔ ژئودزیک‌ها برحسب پارامتر طبیعی و معادلهٔ پارامتری خطوط مستقیم در R^n به‌دست می‌دهد.

قضیهٔ اصلی علاوه براینکه این امکان را بما می‌دهد که ساختار طبیعی قوی را جایگزین مفهوم الصاق آفین کنیم، بیان می‌کند که شعاعهای ساختار طبیعی همان ژئودزیک‌های الصاق آفین هستند، و به‌جای اینکه معادلهٔ ژئودزیک‌ها نسبت به‌دستگاه مختصات موضعی به‌وسیلهٔ دستگاه معادلات دیفرانسیل مرتبهٔ‌دوم داده شوند، مستقیماً به‌وسیلهٔ معادلات خطی به‌دست می‌آیند. همچنین ساختارهای طبیعی این امکان را بما می‌دهند که حداقل قسمت عمده‌ای از هندسهٔ دیفرانسیل را از این طریق به دست آوریم بدون اینکه نیازی به استفاده از تانژانت باندل و یا تنسورها باشد.

لازم به‌ذکر است‌که در این مقاله آنچه از مطالب کلاسیک که مورد نیاز بوده تحت عنوان مقدمه در پاراگرافهای از a تا e بطور خلاصه آورده شده است، و برای تفصیل این مقدمه وسایر یادآوریهای دیگر خواننده می‌تواند به‌یک یا چند کتاب کلاسیک مشهور که در فهرست مراجع گنجانده شده است مراجعه کند. اما در مورد مطالب اصلی، که برای اولین‌بار در این مقاله آمده، تحت شماره‌های از ۱ تا ۱۴ ذکر شده که هر قسمت خود حاوی مطالبی است‌که می‌توانست تحت عنوان یک تعریف یالم ذکرگردد، اما بجز تعاریف مهم و قضیهٔ اصلی، بقیهٔ مطالب بدون عنوان نوشته شده است.

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Normal Structures on Manifolds

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Abstract

Many Structures on a topological m- manifold M may be defined by means of an atlas of local coordinate systems for which the coordinate systems belong to some pseudogroup P of transformations in the model space R^m.

To any symmetric affine connection ∇ on M there is associated a family of normal coordinate systems in a canonical way, via the exponential map. However, the coordinate transformations that occur within this family do not form a pseudogroup of transformations in R^m. On the other hand , normal coordinate systems are «abundant» in the sense that there is at least one such system based at every point of M.

The purpose of this paper is to modify the pseudogroup notion of structure to obtain a characterization of symmetric affine connections.

Preliminaries

We begin this paper by some preliminary remarks that we need.

a. For simplicity I consider the case of a C^{∞}– manifold M modelled on real linear m - space R^m. I suppose that M is connected and ∂M is empty, referring to such an M as a smooth m-manifold.

b. I denote the ring of all smooth $=C^{\infty}$ real - valued functions on M by F(M). Thus the set V(M) of smooth vector fields on M is a linear space over the field R, and a module over the ring F(M).

c. Suppose then that ∇ is a smooth affine connection on the smooth m–manifold M, and let $\xi : U \to U'$ be an admissilble chart on M, where U is an open subset of M, and U' an open subset of R^m. Then there is a basis $X_1,\dots,X_m$ for the module $V(M)|U$, given by $X_j=\partial/\partial u_j$ $j = 1, \dots, m$. Thus I can define a set of m^3 smooth functions $\Gamma^k{}_{ij}: U \to R$ by

$$\nabla_{X_i}^{X_j} = \sum_{k=1}^{m} \bar{\Gamma}^k{}_{ij} X_k. \qquad (1)$$

d. Now let $\eta : V \to V'$ be another chart such that $W = U \cap V \neq \varnothing$, and let $Y_1, \dots Y_m$ be the corresponding basis for the module of smooth vector fields on V. Thus $Y_1 = \partial/\partial v_i$, and

$$\nabla_{Y_i}^{Y_j} = \Sigma \bar{\Gamma}^k{}_{ij} Y_k$$

It follows that on W,

$$\nabla_{Y_i}^{Y_i} = \frac{\partial u_\alpha}{\partial v_i} \nabla \frac{\partial}{\partial u_\alpha}\left(\frac{\partial u_\beta}{\partial v_j} \frac{\partial}{\partial u_\beta}\right)$$

$$= \frac{\partial u_\alpha}{\partial v_i}\left[\frac{\partial}{\partial u_\alpha}\left(\frac{\partial u_\beta}{\partial v_j}\right)\frac{\partial}{\partial u_\beta} + \frac{\partial u_\beta}{\partial v_j}\Gamma^\gamma{}_{\alpha\beta}\frac{\partial}{\partial u_\gamma}\right]$$

$$= \frac{\partial u_\alpha}{\partial v_i}\left[\frac{\partial^2 u_\beta}{\partial v_j \partial v_k}\frac{\partial v^k}{\partial u_\alpha} + \frac{\partial u_\gamma}{\partial v_j}\Gamma^\beta{}_{\alpha\gamma}\right]\frac{\partial v_s}{\partial u_\beta}\frac{\partial}{\partial v_s}$$

(the summation symbol Σ is omitted). Hence by choosing a suitable index, we conclude that

$$\Gamma^\gamma{}_{\alpha\beta} = \Sigma_{i,j,k}\frac{\partial u_i}{\partial v_\alpha}\frac{\partial u_j}{\partial v_\beta}\frac{\partial v_\gamma}{\partial u_k}\Gamma^k{}_{ij} + \Sigma_j \frac{\partial^2 u_j}{\partial v_\alpha \partial v_\beta}\frac{\partial v_\gamma}{\partial u_j} \tag{2}$$

Conversely, if M is a smooth manifold such that for any two overlapping charts ξ, η the smooth functions Γ, $\bar{\Gamma}$ satisfy (2) on their common domain W, then we can define $(\nabla | U)X_i$ by (1).

Now we define $\tilde{\nabla}$ on M by $\left(\tilde{\nabla}_X^Y\right)_p = \left(\left(\tilde{\nabla} | U\right)_{X_1}^{Y_1}\right)_p$ $p \in M$, where X, $Y \in V(M)$ and X_1, Y_1 are the restiction of X, Y on U. This determines a smooth connection ∇ on M.

e. The differential equations for a geodesic segment in the local coordinates of some chart $\xi: U \to U'$ on M are

$$\ddot{x}_k = -\sum_{i,j=1}^{m} \Gamma^k{}_{ij}\dot{x}_i\dot{x}_j$$

For each $P \in M$ and each tangent vector $V \in T_pM$, there is a unique maximal geodesic $\gamma : I \to M$ such that $0 \in I$, $\gamma(0) = P$, and $\gamma'(0) = V$. This geodesic will be denoted by γ_v (the point P being indicated by the context).

Results

After these preliminaries we are ready to study the notion of normal trasformation, which is designed to capture the essence of this relationship.

1: For each $x \in R^m$ there is a smooth path $\gamma_x : R \to R^m$ given by $\gamma_x(t) = tx$. If $x = 0$, $\gamma_x(t) = 0$ for all $t \in R$. But for $x \neq 0$, γ_x maps R bijectively onto the 1- dimensional linear subspace L_x of R^m generated by x. It follows that for all $x \in R^m$ and all t, $s \in R$,

$$\gamma_{sx}(t) = t(sx) = (ts)x = \gamma_x(ts) = s\gamma_x(t),$$

and

$$\gamma'_x(t) = x.$$

For $x \neq 0$, γ_x will be called the ray through x.

2: Suppose next that $f : A \to B$ is a smooth ($= C^\infty$) diffeomorphism, where A, B are open sets of R^m. For each $x \in R^m$ put $A_x = L_x \cap A$. Then for $A_x \neq \varnothing$ there is a non-empty open subset I_A of R such that $\gamma_x | I_A$ is a smooth path $\alpha_x : I_A \to A$ with image A_x. This path composes with f to give a smooth path $f_x = f \circ \alpha_x : I_A \to B$. Of course, the image of f_x is not necessarily a subset of any line L_y, and may intersect transversally, or touch, any ray through B.

3: Def. With the above notation, a smooth diffeomorphism $f : A \to B$ between open subsets of R^m is said to be a normal transformation if the following condition is satisfied.

(N) Let $x \in A$ and $y = f(x) \in B$, γ_x be any ray through x, and suppose that γ_y is any ray through y. If $f(A_x)$ touches γ_y at y, then $f(A_x) = B_y$.

Condition (N) states that the image of any ray through A must cross for all the rays in B transversally or must coincide with the image of one of these rays.

4: Condition (N) says nothing about the relation between the parametrizations of Such coincident rays. We therefore formulate a second condition as follows.

5: As in §2, $f : A \to B$ is a smooth diffeomorphism.

(N*) If $x \neq 0$ and $f(A_x) = B_y$, then there exist a, $b \in R$, $a \neq 0$ such that, for all $t \in I_A$, $f(tx) = (at + b)y$.

Thus (N*) requires that the change of parameters on coincident rays in any normal transformation should be affine.

6: Def. If $f: A \to B$ satisfies both (N) and (N*), we say that f is a strong normal trnsformation. It will be useful to record a few examples befor proceeding further.

7. EXAMPLE 1. Let $\theta : R^m \to R^m$ be a linear automorphism, and let A be any non - empty open subset of R^m. Then $\theta | A : A \to \theta(A)$ is a strong normal transformation.

EXAMPLE 2. Let $\lambda : R^m \to R$ be any smooth function which has a local extremum 0 at 0, and no other critical points. Define $\Phi : R^m \to R^m$ by$\Phi(x) = \lambda(x)$ $x \in R^m$. Then for any non-empty open subset A of $R^m/\{0\}$ the diffeomorphism $f = \Phi | A: A \to B$, where $B = \Phi(A)$, is a normal transformation .

EXAMPLE 3. Suppose that $f: A \to B$ is a strong normal transformation such that $0 \in A \cap B$ and $f(0) = 0$ Consider $g = Df(0)^{-1} \circ f$. Then g is a strong normal transforation of the type described in example 2 in which the function λ is constant on each 1-dimensional linear subsace of R^m. Thus λ is constant, and so f is the restriction of a linear automorphism, as in Example 1.

EXAMPLE 4. Let $g : R^2 \to R^2$ be given by $g(x,y) = (x+1, \ (x^2+1)y)$.

Now let $A = \{(x,y) \in R^2 : x > 0\}$ and $B = \{(x,y) \in R^2 : x > 1\}$. We can show that $f = g | A : A \to B$ is a strong normal transformation.

8 : It is natural to ask next whether the set N (m) of all normal transformations in R^m is a pseudogroup. It follows immediately from definition that N(m) satisfies the first four axioms of the pseudogroup of transformations. The same is true of the set N* (m) of all strong normal transformations in R^m. However, the «composition axiom» of pseodograoup is *satisfied* by neither N (m) nor N* (m) . to see this, consider the followiing pair of transformations. Let A denote the open subset of R^2 given by $0 < \theta < \pi/2$. $0 < r < \pi/2$ in polar coordinates, and let $f : A \to A$ be the diffeomorthism given in Polar coordinates by $f(r, \theta) = (\theta, r)$. Then f maps the rays θ = constant to the arcs r = constant. Next. let $g : R^2 \to R^2$ be defined by $g(x,y) = (x - \pi/2, y)$. Then both f and g are (strong) normal transformations. However, $g \circ f$, which maps A diffeomorphically onto the open subset

$$A - \pi/2 = \{ (r \cos(\theta - \pi/2), \ r \sin\theta) : 0 < \theta < \pi/2, \ 0 < r < \pi/2\}$$

of R^2, is not a normal transformation.

9: It would appear, therefore, that we cannot make use of normal transformations to constrruct atlases and structures on manifolds, unelsss we can advise some substitute for ths composition axiom. Now the normal coordinate systems on an affinely connected manifold M not only have the property that all changes of coordinate are (strong) normal transformations, but they are also very numerous: there is a normal coordinate system associated with every point of M. This fact suggests that we might usually formulate a notion of a normal atlas on a smooth m-manifold M in the following way.

10: Def. Let $A = \{\xi_j : U_j \to U'_j \mid j \in J\}$ be a set of charts on M. Then A is a (strong) normal atlas on M iff

(1) for all $x \in M$ there exists $j \in J$ such that $\xi_j(x) = 0$

(2) for all j, $k \in J$ with $U_j \cap U_k = W \neq \varnothing$, the transformation $\Phi_{jk} : \xi_j(W) \to \xi_k(W)$ is a (strong) normal transformation. Two(strong) normal atlases A & B are equivalent iff $A \cup B$ is a (strong) normal atlas. Each equivalent class of (strong) normal structures contains a unique maximal element called a (strong) normal structure. In what folows, our main concern is with strong normal structures. We give one example, however, to show that there are normal atlases which are not strong normal atlases.

EXAMPLE . Consider real projective m-space $M = P_m(R)$ consisting of all equivalence classes $[x]$ of $R^{m+1}/\{0\}$ under the equivalence relation $\sim$ where $x \sim y$ iff, for some non-zero $\lambda \in R$, $x = \lambda y$. Define an open subset U of $P_m(R)$ by $U = \{[x] \in P_m(R) : x_{m+2} \neq 0\}$, and define a homeomorphism $\xi : U \to R^m$ by $\xi([y]) = (z_1, \ldots, z_m) \in R^m$, where $z_j = y_j/y_{m+1}$, $j = 1, \ldots, m$. Now the orthogonal group 0_{m+1} acts transitively on $P_m(R)$ by $\omega \circ [y] = [\omega(y)]$. For each $\omega \in 0_{m+1}$, let $\xi_\omega : U_\omega \to R^m$ be the homeomorphism given by $\xi_\omega(\omega \circ [y]) = \xi([y])$, $y \in U$. Thus $U_\omega = \omega \circ U$. It is easy to check that $\{\xi_\omega : \omega \in 0_{m+1}\}$

is a normal atlas, but not a strong normal atlas, on $P_m(R)$.

We are now approaching the main therorems of this paper which establishes the equivalence between strong normal structures and afine connections.As a preliminar we establish what we mean by the rays of a strong normal structure on M.

11: Def. Suppose that Λ is a normal structure on M. A smooth path $\gamma : J \to M$ is called a ray on M iff for each chart $\xi : U \to U'$ in Λ there is a ray γ_x in R^m for which taking $J\xi = \gamma^{-1}(U)$, for all $t \in J\xi$, $(\xi \circ y)(t) = \gamma_x(t)$ A ray $\gamma : J \to M$ on N is said to be maximal iff there is no ray δ on M such that J is a proper subset of the domain of δ and $\gamma = \delta | J$.

12 : Lemma. Suppose now that Λ is a srong normal structure. Then the parametrization of any ray on M is detemined up to affine transformation in R. That is, if $\gamma : J \to M$ and $\delta : K \to M$ are rays on M with $\gamma(J) = \delta(K)$, then there exist a, b∈R, $a \neq 0$, such that for all $t \in J$, $\gamma(t) = \delta(at + b)$ This follows immediately from our definitions, as does the statement that any two rays $\gamma : J \to M$ and $\delta : K \to M$ on M must either intersect transversally or coincide imagewise along the complement of some open interval in J. in particular, if γ and δ are maximal rays on M, then either they intersect trasversally or their images coincide.

13:MAIN THEOREM(1) Let ∇ be a smooth affine connection on a smooth m-manifold M. Then the family of all normal coordinate systems on M is a strong normal structure on M.

(2) Conversely, if Λ is a strong normal structure on M, there is a smooth affine connection ∇ on M such that the (max imal) rays of Λ on M are the maximal geodesics cf ∇ on M.

PROOF. (1) Let ∇ be a smooth affine connection on an m- manifold M; then for every point P∈M there exists a normal coordinate system $\xi : U \to U'$ with the pole P, i. e. $\xi(p) = 0$, for every non - zero tangent vector $v_p \in T_pM$ there exists a unique maximal geodesic $\gamma : I \to M$ such that $\gamma'(0) = p$, and $\gamma(0) = v$. we call it a maximal geodesic with initial v_p. Now suppose that $\eta : V \to V'$ be another normal coordinate system with the pole q, such that $U \cap V = W \neq \varnothing$. Let $\xi(W) = A$, $\eta(W) = B$, and $m \in W$, $\xi(m) = x$, If $\gamma : I \to M$ and $\delta : J \to M$ are maximal geodesics with initials v_p and α_q which pass through m, then by the existence and uniqueness maximal geodesic, theorem γ and δ intersect transcversally at m or coincide. Then $f = \eta \circ \xi$ maps $\xi(\gamma(I)) = L_x$ onto $\eta(\delta(J)) = L_y$ or intersects it transversally. In the first case, let γ and δ be parametrised by t and s respectively, and let X, Y be vectorfields such that $X(\gamma(t)) = \gamma'(t)$, $Y(\delta(s)) = \delta'(s)$. suppose that $s = \theta(t)$ where $\theta : \gamma^{-1}(W) \to \delta^{-1}(W)$, then $o = \nabla_x{}^x = \nabla_{\theta' y}{}^{\theta' y}$

Consequently $Y\dot{\theta} = \ddot{\theta} = o$ and $\theta(t) = at + b$ for some a, b∈R and $a \neq 0$ therefore $f(tx) = sy = (at + b) f(x)$, i. e. $f : A \to B$ is a strong normal transformation.

(2). To establish the converse statement (2), suppose that Λ is a strong normal structure on M. Then Λ determines a smooth atlas Λ' on T M as follows. Let $\xi : U \to U'$ be a chart in Λ. Then the ξ- coordinaes $x_1, \ldots, x_m$ in U determine a basis $\partial/\partial x_1, \ldots, \partial/\partial x_m$ for the module of smooth vector fields on U. In particular, for each p∈U, $\partial/\partial x_j |p$ is a basis for T_pM. Thus each $V \in T_pM$ can be written uniquely in the form $V = \sum_{j=1}^{m} yj \frac{\partial}{xj}$. we define a chrat $\xi' : \pi_M{}^{-1}(U) \to U' \times R^m$ by $\xi'(V) = (\xi(\pi_M(V)), y)$ $y = (y_1, \ldots, y_m)$. The family $\Lambda' = \{\xi' : \xi \in \Lambda\}$ is a smooth atlas on T M. Suppose now that Z is a smooth vector field on T M, and consider an integral curve $c : S \to TM$ of Z, for some intenterval $S \subset R$. Then c is the canonical lift of a smooth curve $\gamma : S \to M$ if, for each $t \in S$, $(\pi_{TM})(Z(c(t))) = c(t)$.

It is convenient to denote the canonical lift c of γ by γ', and the canonical lift c' of c by γ''. Thus if $\pi_{TM} \circ Z = 1_{TM}$, then $\gamma'' = Z \circ \gamma'$ may be regarded as a second order differential equation on M, and any curve γ in M with $\gamma'' = Z \circ \gamma'$ and $\gamma'(0) = v \in T_{\gamma(0)}M$ is called a solution of this differential equation with inital condition v.

Let us return now to the chart $\xi' : \pi_M^{-1}(U) \to U' \times R^m$, and let $\xi'(V) = (x,y)$, $V \in \pi_M^{-1}(U)$. Then there are smooth vector fields $\partial/\partial x_1, \dots, \partial/\partial x_m$, $\partial/\partial y_1, \dots, \partial/\partial y_m$ in $T(\pi_M^{-1}(U)) = T^2U$, with an obvious abuse of notation .Thus we can write $Z|TU$ in the form

$$Z = \sum_{j=1}^{m} \left(\alpha_i \frac{\partial}{\partial x_i} + \beta_i \frac{\partial}{\partial y_i} \right)$$

for some smooth function $\alpha_i : U' \times R^m \to R$, $\beta_i : U' \times R^m \to R$. It follows from the above remarks, however, that the conditon $T\pi_M \circ Z = 1_{TM}$ is satisfied iff $\alpha_i(V) = y_i$, where $\xi'(V) = (x,y)$. Hence

$$(1) \quad Z(x,y) = \sum_{i=1}^{m} y_i \frac{\partial}{\partial x_i} + \beta_i(x,y) \frac{\partial}{\partial y_i}.$$

We now apply these considerations to the vector field Z on TM obtained from the rays of Λ on M. That is,we consider the second order differential equation $\gamma'' = Z \circ \gamma'$ on M whose solution curves are the rays of Λ. Suppose then that $\gamma : S \to M$ is a ray on M, and for simplicity suppose that $\gamma(S) \subset U$, where $\xi : U \to U'$ is a chart in Λ, and $\gamma(0) = q$, $\xi(p) = 0$. Then for all $t \in S$, $\xi(\gamma(t)) = tw$, for some $w \in R^m$, $w \neq 0$. Hence $\gamma'(t) = (tw, w)$ and $\gamma'(t) = (w,0)$. we conclude, therefore, that

$$Z(0,y) = (y,0) = \sum_{i=1}^{m} y_i \frac{\partial}{\partial x_i} = Z(v), \quad v \in T_pM,$$

$\xi'(v) = (0,y)$. thus, for all $a \in R$, $Z(0,ay) = (ay,0)$, which we may rewrite in coordinate - free form as

$$(ii) \quad Z(av) = a(T_\alpha)(Z(v)),$$

where $\alpha : TM \to TM$ is given by $\alpha(v) = av$.

Equations (i) and (ii) imply that for all $(x,y) \in U' \times R^m$ and all $a \in R$,

$$\beta_i(x,ay) = a^2 \beta_i(x,y).$$

Thus there are smooth functions $\Gamma^i{}_{jk} : U' \to R$ such that

$$(iii) \quad \beta_i(x,y) = \sum_{j,k=1}^{m} \Gamma^i{}_{jk}(x) y_j y_k.$$

In fact, the functions $\Gamma^i{}_{jk}$ are christoffel symbols of the required affine connection ∇ on M in the chart ξ. To see this, consider another chart $\eta : V \to V'$ in Λ with $W = U \cap V \neq \varnothing$. Then on W we have two expressions for Z namely

$$Z(w) = Z(x,y) = \sum_{i=1}^{m} y_i \frac{\partial}{\partial x_i} + \beta_i(x,y) \frac{\partial}{\partial y_i}$$

$$= Z(\bar{x},\bar{y}) = \sum_{i=1}^{m} \bar{y}_i \frac{\partial}{\partial \bar{x}_i} + \bar{\beta}_i(\bar{x},\bar{y}) \frac{\partial}{\partial \bar{y}_i}$$

where $\bar{x} = \Phi(x)$, $\Phi = \eta \circ \xi^{-1}$, $\bar{y} = D\Phi(x)(y)$.

$$-\beta_i(x,y) = \sum_{j,k=1}^{m} \Gamma^i{}_{jk}(x) y_j y_k,$$

$$-\bar{\beta}_i(\bar{x},\bar{y}) = \sum_{j,k=1}^{m} \Gamma^i{}_{jk}(\bar{x}) \bar{y}_j \bar{y}_k.$$

It follows therefore, after a routine calculation, that

$$\bar{\Gamma}^i{}_{jk}(x) = \sum_{\alpha,\delta,\gamma} \frac{\partial x_\alpha}{\partial \bar{x}_j} \frac{\partial x_\delta}{\partial \bar{x}_k} \frac{\partial \bar{x}_i}{\partial x_\gamma} \Gamma^\gamma{}_{\alpha\delta}(x) +$$

$$+ \sum_{\delta} \frac{\partial^2 x_\delta}{\partial \bar{x}_j \partial \bar{x}_k} \frac{\partial \bar{x}_i}{\partial x_\delta}$$

thus, in view of the results of § 4 it follows therefore, the functions $\Gamma^i{}_{jk}$ defined on each chart of Λ are the christoffel symbols of an sffine connectoon ∇ on M, whose geodesics are the rays of Λ.

The main theorem shows that it is possible to define the concept of an affine connection in such a way that the geodesics, that is the rays are given directly by linear equation in local coordinates rather than by a system of second-order differential equations. However, if γ is a solution curve of

$$\text{(iv)} \qquad \gamma''(t) = Z \circ \gamma'$$

which does not pass through the poles of the chart ξ and $\xi \circ \gamma(t) = x(t)$, then we have

$$\text{(v)} \quad \gamma''(t) = \sum_i \left(\frac{dx_i}{dt}\right)\frac{\partial}{\partial x_i} + \sum_i \left(\frac{d^2x_i}{dt^2}\right)\frac{\partial}{\partial y_i}$$

put from (i) and (iii)

$$Z(x,y) = \sum_i \left[y_i \frac{\partial}{\partial x_i} - \left(\sum_{j,k} \Gamma^i{}_{jk}(x) y_j y_k \right) \frac{\partial}{\partial y_i} \right]$$

or

$$\text{(vi)} \quad Z(\dot{\gamma}) = \sum_i \left[\left(\frac{dx_i}{dt}\right)\frac{\partial}{\partial x_i} - \left(\sum_{j,k} \Gamma^i{}_{jk}(x) \frac{dx_j}{dt} \frac{dx_k}{dt} \right) \frac{\partial}{\partial y_i} \right]$$

thus from (v) and (vi) we find that γ is given in ξ-coordinates by the classical second order differential equations

$$\text{(vii)} \quad \frac{d^2x_i}{dt^2} = - \sum_{j,k} \Gamma^i{}_{jk}(x) \frac{dx_j}{dt} \frac{dx_k}{dt} .$$

It should also be noted that the use of strong normal structures does not involve the tangent bundle. On the other hand the mechanism of the normal coordinate systems serves to transfer the linear structure of R^m (and of any linear space) frome the model space R^m in the appropriate affinely connected m-manifold M. The one-dimensional linear subspaces of R^m are mapped to the geodesics in M and most areas of differential geometry can be handled directly in this approach.

References

Bishop R. and Crittenden R., 1970 : *Geometry of Manifolds*, Academic press, New York.

Brickell, F. and Clark R. S., 1970 : *Differentiable Manifolds*, Van Nostrand, New York,

Burke w., 1985 : *Applied Differanttal Geomerty*, Cambridge University Press, Cambridge.

Crampin, -M., Pirani, F. A. E., 1986 : *Applicable differentiable Geometry* , Cambridge unifversity press,

Martin , George E., 1982 : *Transfromation Geometry, an introduction to symmetry*, Springer Verlag , Berlin.