معادلات از نوع لورنتیو غیر یکنواخت و مسائل با مقادیر مرزی

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چکیده
یک نتیجه منتشره شده قبلی گسترش داده شده است . منظور اصلی در این تحقیق پرکردن رخنه بین دستگاه‌های معادلات خطی وغیر خطی با دو مجهول در حوزه‌های صفحه‌ای می‌باشد. مسئله با مقادیر مرزی ازنوع ریمن- هیلبرت وهیلبرت برای دستگاه معادلات ، بیضوی به مفهوم لورنتیو مورد بررسی قرارگرفته‌است.

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Non - Uniformly Lavrentiev Type Equations And Boundary Values Problems

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Abstract

An earlier reported reault is extended. The main purpose of this work is to fill the gap between the linear and nonlinear systems of equations with two unknowns in the plane domains. the boundary values' problem of Riemann - Hilbert and Hilbert type for the system of equations, elliptic in the sense of Lavrentive has been studied.

1. Introduction

In the present work, attention will be confined to the boundary values problem for the non-linear system of two first order real equations with degeneration of ellipticity. As it is well - Known, complex variable methods provide an effective means for investiga - ting some classes of partial differential equations . They form the basis for the so - called function theoretic methods (see for instance [31]) which, though used specially in the study of equatins with two independent variables, can be applied to some cases with several independent veriables (see for example [71]).

Our mean is to exhibit some representation for the solution of the problem in the case of non-uniformly ellipticity of the equation. It will be indicated that under some natural smoothness and convexity conditions, these problems are well - posed. None of these will be treated in depth, the intent being to give the solution of the problem in the case of negative index corresponding to the boundary values problem.

Let us consider a class of general non - linear system of two first order real squations.

$$\psi_i\,(x, y, u, v, u_x, u_y, v_x, v_y) = 0 \quad , \quad i=1, 2 \quad (1.1)$$

with two unknowns u and v of the independent variabes x, y, which can be written in the complex form

$$w_{\bar z} = H\,(z, w, w_z), \qquad (1.2)$$

(see also [2] or [7]) where $z = x + iy$, $w = w(z) = u(x,y) + iv(x,y)$ and all derivatives $\partial u/\partial x = u_x, \ldots, \partial v/\partial y = v_y$ are stated by the derivatives

$$\partial w/\partial z = w_z,\ \partial w/\partial \bar z = w_{\bar z}\,.$$

Clearly, equation (1.2) contains the complex form

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of the Cauchy - Riemann system $W_{\bar{z}}=0$.

Let L be the boundary contours of a Liapounoff region G and $\hat{L}$ be another system of finite non- intersecting contours inside G which decomposes G into a finite number of regions, the union of all these subsets of the domain G will be called the domain $\hat{G}$; L and $\hat{L}$ have no common points.

Consider the equation (1. 2) in the domain $\hat{G}$, fulfilling the boundary conditions.

$$\mathrm{Re}\,[\,\overline{a(t)}\,w(t)\,]=\gamma(t) \tag{1. 3}$$

on L, and

$$w^{+}(t)=g(t)w^{-}(t)+h(t) \tag{1. 4}$$

on $\hat{L}$. The symbols w^{+} and w^{-} are understood in the usual sense of the theory of the Hilbert boundary values problem , a, γ and g, h are given functions on L and $\hat{L}$ respectively.

Let us assume that the function $H(\zeta) = H(z, w, \zeta)$ fulfils the Lipschitz condition.

$$|H(z, w, \zeta_1) - H(z, w, \zeta_2)| \leq q(z, w)\,|\zeta_1 - \zeta_2| \tag{1. 5}$$

$$q(z, w) \leq q_0 < 1. \tag{1. 6}$$

Then, equation (1. 2) satisfying the conditions (1.5), (1. 6) is called uniformly elliptic in the sense of Lavrentiev in the given domain (see also [2]).

Under some natural assumptions on the coefficientes (A) : Hölder continuity of a, γ and g, h on the boundary L and $\hat{L}$ respectively; $\hat{L}$ belonging to class C^1 and the solution being sought in the class of sectionally continuous functions in $\hat{G}$, which have continuous extensions up to the boundary and belonging to the class W^1_p, $p > 2$, it can proved that:

Lemma 1. Boundary values problem (1. 2), (1. 3), (1. 4) for (1.2) in $\hat{G}$ is equivalent to a boundary values of the type (1. 2), (1. 3) in G. To avoid a long expression, we shallnot bring here the prove, see for instance [5].

Remark 1.Generally, the exponent é of the Hölder continuity of g, h on $\hat{L}$ will be assumed to be $1/2 <$ é < 1, but in the case of $H = 0$ ($w_{\bar{z}} = 0$) or a larger class of functions , i. e. in the case of generalized analytic functions, é may be assumed to be $0 <$ é < 1.

2. **A Non-Uniformly Elliptic Boundary Values Problem**. In this sectoion, we shall bring an extension to Lavrentiev's condition (see also [6]), for which the boundary values problem (1. 2), (1.3) in some cases be solvable. Therefore instead of q in (1.5), (1. 6) which assumed to be a real function of complex variables z,w, suppose that q be a real function of complex variables z , w , ζ_1 , ζ_2 , in view of this hypothesis, the function q (z, w, ζ_1 , ζ_2) satiafies the following inequalities

$$|H(z,w, \zeta_1) - H(z, w, \zeta_2)| \leq q(z, w, \zeta_1, \zeta_2)\,|\zeta_1 - \zeta_2| \tag{2.1}$$

$$q(z, w, \zeta_1, \zeta_2) \leq 1 \tag{2. 2}$$

It is clear that (2.1), (2. 2) are not enough for investigating the boundry values problems and some supplemental conditions are necessary. Henceforth, let us consider the equation

$$w_{\bar{z}} = \hat{H}(\mu(z)\,w_z) \tag{2. 3}$$

where $\hat{H}$ as a comlex function fulfils the following condition

$$|\hat{H}(\zeta_1) - \hat{H}(\zeta_2)| \leq \hat{q}(|\zeta_1 - \zeta_2|)\,|\zeta_1 - \zeta_2|, \tag{2. 4}$$

where $\hat{q}$ as a function à is continuous in $[0, \infty]$ (à$=|\zeta_1 - \zeta_2|$) ;$\hat{q}$(à) < 1 for à $\varepsilon[0,\infty]$; the function à. $\hat{q}^2$(à) is increasing and concave; the complex function $\mu(z)$ is assumed to be measueable belonging to $L_\infty(G)$.

The number

$$\hat{q}_0 = \limsup_{á \to \infty} (\hat{q}(á)) < 1 , \tag{2.5}$$

is called the coefficient of ellipticity corresponding to the boundary values problem (2.3), (1.3).

Let a (t)-the coefficient relative to the boundary condition (1.3) does not vanish on L, then the index corresponding to the boundary values problem (2. 3) - (1.3) is defined as follows

$$n = \mathrm{ind}\ a = 1/2\pi i \int_L d(\log a(t))$$

Remark 2. It is well-known that, in the classical boundary values problems of the type (1.3), relative to the equation (1.2) with uniformly ellipticity, the solution w is sought in the Sobolev space $W_p^1(G)$, for some $p>2$. for the equation (2. 3) in the case of non-uniformly ellipticity (2. 4) we shall not apply the L_p - theory directly for the proof of the existance, therefore the formulation of the boundary values problem (2.3), (1.3) involve the weak boundary condition (see also [6]).

Let us assume G be a simply connected domain of Liapounoff type $|\mu(z)|\leq 1$, then we have:

Proposition 1. If the index $n<0$, then there exists a solution (in W_p^1 for some $p>2$) of the boundary values problem (2.3), (1.3).

We ahall not bring here the prove, since this involve extremely lengthy expression, it can be carried out similar to [6].

Proposition 2. For negative index n, the solution w of the boundary values problem (2.3), (1.3), belonging to $W_p^1(G)$, $p>2$ is unique.

Proof : Making use of the representation formula $w=\hat{T}(\omega)$ (for explicit form of $\hat{T}$ see [1]), where $\hat{T}(\omega)=(\hat{T}(\omega))_G$, $\omega\in L_p(G)$, $p\geq 2$, we observe that the operator $\hat{T}$ satisfies the boundary condition (1.3) on L and $\partial\hat{T}(\omega)/\partial\bar{z}=\omega(z)$ in G ($\hat{T}$ depends on the index n). Denoting by $\hat{S}\omega=\hat{S}(\omega)=\partial\hat{T}(\omega)\,\partial z$, we conclude that the L_2 - norm of $\hat{S}$ is equal to one, sinc $n>0$, also $\hat{S}$ is a bounded operator from $L_p(G)$, $p>1$ into itself and the continuity of $\|\hat{S}\|_p$ with respect to $p>1$ can be proved through the well - known Riesz -Thorin convexity theorem. By concavity property of $\acute{a}\hat{p}^2(\acute{a})$ and Jenssen inequality, we conclude that if $w_1=\hat{T}(\omega_1)$, $w_2=\hat{T}(\omega_2)$ are Solutions of the boundary values problem (2.3), (1.3), then $\omega_1=\omega_2$ a. e. in G but in view of the of properties $\hat{T}$, we obtain $w_1=w_2$, which proves the uniqueness. Under some refinements, this method can be extended to the multiply connected domains.

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