اصلاحی بر حرکت روی منحنی در روش هوموتوپی پیوسته

دکتر محمود محسنی‌مقدم
بخش ریاضی دانشگاه شمالی ایلی نویز، آمریکا و بخش ریاضی دانشگاه‌کرمان، ایران

چکیده

پیدا نمودن جوابهای معادلات تحلیلی از قدیم الایام مورد توجه و بحث ریاضی‌دانان بوده و روشها مختلفی برای حل اینگونه مسائل ارائه شده است. در این مورد از اوائل دهه ۱۹۳۰ روشی بنام هوموتوپی ارائه شده‌که هنوزهم زمینه بحث فراوان دارد. اصول این روش به اختصار چنین است: فرض کنیم F(x) معادله‌ای است‌که در شرایط خاصی صدق می‌کند. برای پیدا نمودن صفرهای این معادله‌ابتدا معادله‌ای ارائه می‌شود‌که صفرهای آن به‌سهولت قابل محاسبه است و به‌طریقی درارتباط با F(x) می‌باشد. سپس یک هوموتوپی بین F(x) و معادله اخیربرقرار می‌نمائیم. از این مرحله به بعدروش‌کار به‌این صورت است که از جوابهای معادله اخیر بکمک هوموتوپی داده شده‌بطور پیوسته‌حرکت نموده و انتظار می‌رود که نهایتاً به جوابهای F(x) برسیم. اما چگونگی تعریف هوموتوپی مورد بحث، تعیین معادله‌ای که جوابهای آن به سهولت قابل محاسبه‌اند، چگونگی حرکت بطور پیوسته، امکان حرکت، و نهایتاً راههای مقرون به صرفه چنین حرکتی متضمن مسائل و مشکلات عدیده‌ای می‌باشد. در این مقاله سعی شده است که جهت حرکت روی منحنی در هرلحظه تعیین شود. بعبارت دیگر ثابت شده است که این جهت در ارتباط با حاصلضرب میدان برداری جوابهای معادله دیفرانسیل متناظر با هوموتوپی مفروض پیدا می‌شود.

An improvement on the path following technique of the homotopy continuation method

Dr. Mahmoud Mohseni - Moghadam

Department of Mathematical sciences Northern Illinois vniversity U. S. A.

and

Mathematics Department, University of Kerman, IRAN

Abstract

In order to approximate the solution of an analytic function by homotopy continuation method, one follows a curve starting from a trivial system and numerically moves along this curve to find the solution of the given function.

In this paper a new technique is developed to determine the orientation of the movement along the path by computing of the vector field of the corresponding differential equation. Namely as a main result we show that the orientation is derived from computation of the corresponding vector field.

1. Inrtoduction

Let F be a smooth function from $\mathbf{R}^n$ to $\mathbf{R}^n$. In this paper we shall first consider the problem of finding the solution of $F(x)=0$, by homotopy continuation method. The term «continuation method» is derived from a class of numerical methods dating at least back to Lahaye (1934 & 1935), and also known as «embedding method»., Detailed discussion of these methods can be found in articles by Wacker (1978), and Allgower & George (1980). One starts with a trivial equation, cne to which the solution is obvious and immediately known Then the system is deformed continuously to $F(x)=0$. In general, the solurion of the trivial system will prescribe under this deformation, a smooth curve which is conne eted to the solution of $F(x)=0$. Our discussion hers is limited to following the curve. As a main result, we show the determination of the orientation of the curve is a by - product of the computation of the vector field of the ordinary differential eqation.

2. Homotopy and path existence

In early 1950's, Davidenko (1953a & b) introduced a method of solving $F(x)=0$. where F is a smooth - function from $\mathbf{R}^n$ to $\mathbf{R}^n$. Let $H : \mathbf{R}^n \times [o, 1] \rightarrow \mathbf{R}^n$ be defined as

$$H(x, t)=(1 - t)\ (x - a)+tF(x) \qquad (2.1)$$

with $a \in \mathbf{R}^n$ given. It is clear that $H(x, 0)=x - a$ and $H(x, 1)=F(x)$, Suppose.

(A): the partial derivarive H is always nonsingular. Then by application of the Implicit Function Theorem, there exists a curve x(t), as a function of t, such that

$$H(x(t),t)=0. \qquad (2.2)$$

we differentiate (2. 2) with respect to t, to get the differential equation

$$\frac{dx}{dt}=-H^{-1}{}_x H_t, \qquad (2.3)$$

$$x(0)=a.$$

Thus, finding a zero of F(x) is equivalent to solving the initial value problem(2. 3), and finding its value at t=1. The assumption (A) is rather strong, and hence the power of Davidenko's merthod is restricted.

Let us consider the homotopy

$$H : \mathbf{R}^n \times \mathbf{R}^n \times (0, 1) \rightarrow \mathbf{R}^n \qquad (2.4)$$

defined by

$$H(x, a, t) = (1 - t)\ (x - a) + tF(x)$$

with $x \in \mathbf{R}^n$, $a \in \mathbf{R}^n$, $t \in (0, 1)$. For a fixed $a \in \mathbf{R}^n$, define

$$H\ : \mathbf{R}^n \times (0, 1) \rightarrow \mathbf{R}^n \qquad (2.5)$$

by $H_a(x, t)=H(x, a, t)$.

The following thorem may be found in Transversal mapping and flows, by Abraham and Robbin (1967).

Theorem 2. 1 (Generalized Sard's Theorem)

Let $V \subset \mathbf{R}^n$, $W \subset \mathbf{R}^m$ be open and let

$$G : V \times W \rightarrow \mathbf{R}^p$$

be smooth If $o \in \mathbf{R}^p$ is a regular value for G, then for almost every $a \in V$ (int he sense of either Baire category or Lebesgue measure), 0 is a regular value for $G_a(.) \equiv G(a,.)$.

For our homotopy defined in (2. 4) we have the following.

Lemma 2. 2

For almost every $a \in \mathbf{R}$, zero is a regular value of

$$H_a : \mathbf{R}^n \times (0.\ 1) \rightarrow \mathbf{R}^n$$

Where H_a is given in (2. 5) .

The overall idea is to start from a trivial solution of $H_a(.,0)=0$, and follow the path generated in $H_a(.,t)$ as t goes from zero to one. we hope the trivial solution deforms into the solution of the original sysetm, and hence we would be able to follow the connected path from the trivial system to the solution of $F(x)=0$.

of course this is quite an idealized process, and there are a number of difficulties. First of all, in general, a path need not exist. Second, if one exists, it might be very ill - behaved. In other words the set

$$\{(x.\ t) : x \in \mathbf{R}^n,\ t \in (0, 1),\ H_a(x.\ t)=0\}$$

may consist of different solutions, such as isolated points, self - convergings, bifurcations, endless spirals, closed orbit, and smooth paths. But we are interested only in smooth paths,

Let a be choosen so that 0 is a regular value for $H_a(x. t)$ (because of Lemma (2. 2) this can be done with probability one). Then repeated use of the Implicit, function Theorem implies that $H_a^{-1}(0)$ consists of one dimensional manifolds. Detailed discussion for the existence of paths are given by Garcia and Zangwill (1978), Garcia (1979a & b) and Chow, Mallet - Paret and Yorke (1978). Let Γ_a be the component of $H_a^{-1}(0)$ with a as one endpoint. Also let us assume this component is parameterized by s. For notational convenience we refer to $H_a(x. t)$ by $H(x. t)$. Therefore,

$$H(x(s),\ t(s))=0,$$
$$x(0)=a. \qquad (2.\ 6)$$

Differentiation of H with respect to parameter s yields

$$H_x(x(s),\ t(s)).\ \dot{x}+H_t(x(s),\ t(s))\dot{t}=O.$$
$$x(0)=a. \qquad (2.\ 7)$$

Here H_x and H_t are respectively the partial derivative of H with respect to x and t. The ordinary differential equations (2. 7) can be written in the following matrix form

$$[H_x\ H_t]\begin{bmatrix}\dot{x}\\ \dot{t}\end{bmatrix}=0,$$
$$\begin{bmatrix}x(0)\\ t(0)\end{bmatrix}=\begin{bmatrix}a\\ 0\end{bmatrix}. \qquad (2.\ 8)$$

The integral curve of this differential equation, namely $(x(s),\ t(s))$ is a simple curve starting form (a. 0). In the next section we carefully examine the movement along this curve.

3. Movement along the Path

We have seen that $H_a^{-1}(0)$ consists of only arcs and closed curves. These curves are the solutions of the ordinary differential equations

$$[H_x\ H_t]\begin{bmatrix}\dot{x}\\ \dot{t}\end{bmatrix}=0,$$
$$\begin{bmatrix}x(0)\\ t(0)\end{bmatrix}=\begin{bmatrix}a\\ 0\end{bmatrix}, \qquad (3.\ 1)$$

where H_x is an $n\times n$ matrix, H_t is an $n\times 1$ matrix and $\cdot=d/ds$ for some paramerer s. For the remainder of this section, we will let s be the arc length. since 0 is a regular value of H, $[H_x\ H_t]$ is of full rank. Hence the kernel of $[H_x\ H_t]$ is one - dimensional, by above the vector $[\dot{x},\ \dot{t}]$ lies in this kernel.

Let $A=[H_x\ H_t]$ and $y=\begin{bmatrix}x\\ t\end{bmatrix}$. Then (3. 1) simplifies to

$$A\dot{y}=O,\ y(0)=\begin{bmatrix}a\\ 0\end{bmatrix}. \qquad (3.2)$$

with $\|\dot{y}\|_2=1$, where $\|\ \|_2$ is the ordinary Euclidean norm. The equation $A\dot{y}=0$ means that $\dot{y}$ is perpendicular to the row space of A. Let $A^t = QR$ where Q is an $(n+1)\times(n+1)$ orthogonal matrix and R is an $(n+1)\times n$ upper triangular matrix with $r_{ii}\geqslant 0$ $(i=1,\dots,n)$.

Suppose for some s, $\dot{y}(s)$ is known, hence A is known. The following lemma enables us to find $\dot{y}(s+\Delta s)$ and trace the path by an ordinary differential equation solver.

Lemma 3. 1

Let q_{n+1} be the last column of the orrhogonal matrix Q, then

$$\dot{y}=\pm q_{n+1}. \qquad (3.\ 3)$$

Proof :

Since $A^t = QR$, and $A\dot{y} = 0$, we have

$$R^t Q^t \dot{y}=0.$$

Since matrix R has rank n, we get

$$r_{ii}\neq 0\quad i=1,\ 2,\dots,\ n.$$

Suppose $Q^t \dot{y} = (\beta_1,\ \beta_2,\dots,\ \beta_{n+1})^t$, then we have

$$r_{11}\beta_1=0$$
$$r_{12}\beta_1+r_{22}\beta_2=0$$
$$\vdots$$
$$r_{1n}\beta_1+r_{2n}\beta_2+\cdots+r_{nn}\beta_n=0.$$

Because of (3. 4) this system implies

$$\beta_1=\beta_2=\cdots=\beta_n=0.$$

Hence

$$Qty=(0,0,\cdots,\beta_{n+1})^t.$$

So

$$\dot{y}=Q(0,0,\cdots,\beta_{n+1}).$$

Therefore $\dot{y}$ is a scalar multiple of the last column of Q.

Since $\|\dot{y}\|_2=1$, we get

$$\dot{y}=\pm q_{n+1}.$$

In order to determine the orientation of $\dot{y}$, we give the following theorem which can be found in Garcia and Gould (1978).

Theorem 3. 2

Let $H:\mathbf{R}^{n+1}\rightarrow R^n$ be a C^1 map, and let $z(s)=(z_1(s),\cdots,z_{n+1}((s))$ be a C^1 curve in $\mathbf{R}^{n+1}$ satisfying

$$H(z(s))=0.$$

Then for all s either

$$\text{sgn } z'_i(s)=\text{sgn det } H^i(z(s)) \qquad (3.5)$$

or

$$\text{sgn } z'_i(s)=-\text{sgn det } H^i(z(s)) \qquad (3.5.1)$$

where $z'_i(s)=\dfrac{dz}{ds}$ and H^i is the Jacobian of H with ith column deleted.

Applying this theorem to our homotopy, we get either

$$\text{sgn } \dot{t}(s)=\text{sgn det } H_x(x(s),t(s)) \qquad (3.6)$$

or

$$\text{sgn } \dot{t}(s)=-\text{sgn det } H_x(x(s),t(s)) \qquad (3.6.1)$$

for all s. However at $s=0$. H_x is the $n\times n$ identity matrix and hence det H =1. We may assume$t(0)>0$, therefore(3.6)holds for all s. Thus sgn $\dot{y}$ in (3. 3). isdeter minde as soon as we know sgn $\dot{t}(s)$, and to this end we prove the following proposition :

Proposition 3. 3

Let $Q=(q_{ij})$, then

$$\text{sgn } \dot{t}(s)=(-1)^n \text{ sgn } (q_{n+1},q_{n+1}), \qquad (3.7)$$

Proof

Let $e_{n+1}\in\mathbf{R}^{n+1}$ be the (n+1) th unit vector . That is

$$e_{n+1}=(0,0,\cdots,1)$$

then

$$Q^t[A^t e_{n+1}]=Q^t[QR \quad e_{n+1}]=[R \quad Q^t e_{n+1}].$$

By property of the Houscholder transformations (For detail see Raleston, 1978)

$$\det Q^t=\det p_n \det p_{n-1}\cdots\det p_1=(-1)^n.$$

Where $P_1,\cdots,p_n$ is a sequence of Houscholder tran - sformations such that

$$P_n P_{n+1},\cdots,P_1 A^t=R.$$

Hence

$$\text{der }[R \; Q^t e_{n+1}]=\det Q^t.\det[A^t e_{n+1}]$$
$$=(-1)^n\det\begin{bmatrix}H^t{}_x & 0\\ H^t{}_t & 0\end{bmatrix}=(-1)^n\det H_x{}^t.$$

On the orher hand, since R is an upper triangular - matrix with $r_{ii}>0$, we have

$$\text{sgn det }[R \; Q^t e_{n+1}]=\text{sng}(q_{n+1,n+1}). \qquad (3.8)$$

Therevore

$$\text{sgn } \dot{t}=\text{sgn det } H_x{}^t=(-1)^n \text{ sgn }(q_{n+1,n+1}) \quad (3.9)$$

From the above discussion we see that in order to follow the curve Γ_a, namely, finding the solution of homotopy equation (2. 6) at t=1 can be summarized as follows : We start at t=O and compute H_x, H_t and factorize

$$A^t=\begin{bmatrix}H^t{}_x\\ H^t{}_t\end{bmatrix}$$

as a product oi an $(n+1)\times(n+1)$ orthogonal matrix $Q=(q_{ij})$ and an $(n+1)\times n$ upper triangular matrix R. Then $\begin{bmatrix}\dot{x}(s)\\ \dot{t}(s)\end{bmatrix}$ is given by the last column of Q with a possible sign change, and the sign of this - vector is given by (3. 7). Next we solve the differential equation (2. 8) and continue this process until t=1.

References

Abraham,R. and Robbin, J. (1967). *Transversal mappong* and flows Benjamin, New York.

Allgower, E. and Georg, K, (1980). Simplicial and continuation methods for approximating fixed - points and solutions to systems of equations *SIAM Review*, **22(1)**, 28-85

Chow, S. N., Mallet paret, J. and Yorke, J. A. (1978). Finding zeros of maps; Homotopy methods that are constructive with probability one *Math. Comput.*, **32**, 887 - 899.

Davidenko, D. (1953 a), on a new method of numeri - cally integrationg a system of nonlinear equations. *Dokl, Akad. Nauk SSSR*, **88**, 601 - 604. (in Russian)

Davidenko, (1953b), on the approximation solution of a system of nonlinear equations, *Ukrain, Math. Z.* **15**, 190 - 206 (in Russian).

Garcia, C. B. and Gould, F. J. (1978). A theorem on homotopy paths, Mathematics of operational research. **3 (4)**, 282-289.

Garcia, C. B. and Zangwill,W. I. (1978). Global continuation methods for finding all solutions to polynomial systems of equations in n variables, in : *Symposium on external methods and systems analysis* . Springer Verlag, Lerlin.

Garcia, C. B. (1979a). Determining all solutions to systems of nonliuear equations. *Mathematics of operations Research.* **4(1) 1-14**.

Garcia, C. B. (1979b). An approach to homotopy and degree theory *Mathematies of operations Research* . **4.44**, 390 - 405.

Lahaye, E. (1934). Une methode de resolution d'une cate'gorie d'equations trancendantes, C. R. *Acad, Sci.*, Paris, **198**, 1840 - 1842.

Lahaye, E. (1935). Sur la representation des pracines systemes d'equations transcendantes, *Deoxiome Congres National de Science*, **1**, 141 - 146.

Raleston, A. and Rabinowitz, P. (1978). *First course in numerical analysis*. McGraw - Hill book company, New York.

Walker, H., ed. (1978). *Continuation methods*. Academic Academic Press, New York.