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دربارهٔ فرمول باز برای توابع موجی هیپرژئومتریک ورادیال

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خلاصه*

دوفرمول باز شده برای توابع فریه (Fériet) وکامپ (Kampé) ارزش یابی شده‌است بااستفاده ازخواص اورتوگونال توابع بسل (Bessel) چندمورد جالب همراه باکاربرد یکی از آن‌ها درتوابع موجی رادیال برای اتم های شبه هیدرژن نیز مورد بحث قرارگرفته شده است.

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yield some very interesting resuls. This fact is established in the light of the results [10, p. 105, 106]

$$ {}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}b\,,\ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}\,; \\ a, \qquad a+b-1\,;\end{matrix}\ 4x\right]={}_0F_1\left[\begin{matrix}-\,; \\ a\,;\end{matrix}\ x\right]{}_0F_1\left[\begin{matrix}-\,; \\ b\,;\end{matrix}\ x\right] $$

and

$$ {}_2F_3\left[\begin{matrix}a\,,\ b-a\,; \\ b\,,\frac{1}{2}b\,;\ \frac{1}{2}b+\frac{1}{2}\,;\end{matrix}\ \frac{1}{4}x^2\right]={}_1F_1\left[\begin{matrix}a\,; \\ b\,;\end{matrix}\ x\right]{}_1F_1\left[\begin{matrix}a\,; \\ b\,;\end{matrix}\ -x\right] $$

Since by proper choice of parameters ${}_0F_1$ can be reduced to Bessel function and can also be transformed to ${}_1F_1$ by Kummer's second therorem [10, p. 126]. Further ${}_1F_1$ can be reduced to Whittaker function $M_{k,m}(x)$, generalized Laguerre polynomial $L_n^{\alpha}(x)$. Hermite polynomial $H_n(x)$ and regular and irregular coulomb wave functions F_L and G_L: thereby providing us with such results as may be useful in various problems encountered in Quantum mechanics viz., Collision prodlem of two particles with Coulomb interaction, Harmonic oscillator and the Hydrogen atom [6, pp. 491, 415, 421].

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$$R_{nl}(\gamma) = -\left(\sqrt{\frac{z}{n^2 a_o} \cdot \frac{(n+l)!}{(n-l-1)!}}\right) \frac{e^{-\frac{z\gamma}{na_o}}}{(2l+1)!} \gamma^{l} \left(\frac{2z}{na_o}\right)^{l+1}$$

$$\left\{ 2 \sum_{s=o}^{\infty} J_v(\mp p\sqrt{\gamma})\, {}_3F_1 \left[\begin{matrix} -n+l+1; \\ \pm \frac{v}{2}, \\ 2l+2; \end{matrix} \; \frac{-8z}{np^2 a_o} \right] \right\} \tag{5·4}$$

which is valid under the same conditions as given for (5.3) with $\rho = O$.

Although the Hydrogen like radial wave functions appear to be very complicated, they actually reduce to relatively simple forms, especially for low values of total quantum number n and azimuthal quantum number l. The expressions for $R_{nl}(\gamma)$ computed from (5.4) for $n=1$ and 2, for $l=1$ and O are given in the table given below :

NORMALIZED RADIAL FUNCTIONS FOR HYDROGEN LIKE ATOMS

n	l	Rnl(γ)
1	0	$2\left(\frac{z}{a_o}\right)^{\frac{3}{2}} e^{\frac{-z\gamma}{a_o}} \left\{2 \sum_{s=o}^{\infty} J_v(\pm p\sqrt{\gamma})\right\}$
2	0	$\frac{1}{\sqrt{2}}\left(\frac{z}{a_o}\right)^{\frac{3}{2}} e^{\frac{-z\gamma}{2a_o}} \left\{2 \sum_{s=o}^{\infty} J_v(\pm p\sqrt{\gamma})\left(1 - \frac{v^2 z}{2a_o p^2}\right)\right\}$
2	1	$\frac{1}{2\sqrt{6}}\left(\frac{z}{a_o}\right)^{\frac{3}{2}} \left(\frac{z\gamma}{a_o}\right) e^{\frac{-z\gamma}{2a_o}} \left\{2 \sum_{s=o}^{\infty} J_v(\pm p\sqrt{\gamma})\right\}$

The expressions for $R_{nl}(\gamma)$ for different values of quantum numbers n and l obviously involve here a series in Bessel functions $J_v(\pm p\sqrt{\gamma})$.

6. CONCLUSIONS

We conclude here with the remarks that on reducing the generalized hypergeometric functions on the L.H.S. of expansions (4.1), (4.2), (4.3) and (4.4)

5. RADIAL WAVE FUNCTIONS:

In this section, we make use of the particular case obtained in equation (4.3) in the radial wave functions for Hydrogen like atoms.

We know that [9, p. 132, (21-4)] the normalized radial part of the wave function for the Hydrogen atom is

$$R_{nl}(\gamma) = -\left[\left(\frac{2z}{na_o}\right)^3 \frac{(n-l-1)!}{2n\left\{(n+l)!\right\}^3}\right]^{\frac{1}{2}} e^{-\frac{z\gamma}{na_o}} \left(\frac{2z\gamma}{na_o}\right)^l L_{n+l}^{2l+1}\left(\frac{2z\gamma}{na_o}\right) \tag{5.1}$$

Changing the associated Laguerre function in (5.1) into the confluent hypergeometric function [12, p. 166, (44.3], we get

$$R_{nl}(\gamma) =$$

$$-\left(\sqrt{\frac{z}{n^2a_o}\left(\frac{1}{\gamma^2}\right)\frac{(n+l)!}{(n-l-1)!}}\right)\frac{e^{-\frac{z\gamma}{na_o}}}{(2l+1)!}\left(\frac{2z\gamma}{na_o}\right)^{l+1} {}_1F_1\left(-n+l+1;\ 2l+2;\ \frac{2z\gamma}{na_o}\right) \tag{5.2}$$

Now setting $C=D=1$, $c=-n+l+1$, $d_1=2l+2$; replacing t^2 by γ and $x=\dfrac{2z}{na_o}$ in equation (4.3), we get finally with the help of (5.2):

$$R_{nl}(\gamma) = -\left(\sqrt{\frac{z}{n^2a_o}\cdot\frac{(n+l)!}{(n-l-1)!}}\right)\frac{e^{-\frac{z\gamma}{na_o}}}{(2l+1)!}\,\gamma^{l-\rho}\left(\frac{2z}{na_o}\right)^{l+1}$$
$$\times \sum_{s=o}^{\infty} \frac{2^{2\rho}\,\Gamma\left(\rho+\frac{\nu}{2}\right)\nu J_\nu(\pm p\sqrt{\gamma})}{p^{2\rho}\,\Gamma\left(1+\frac{\nu}{2}-\rho\right)}\ {}_3F_1\left[\begin{matrix}\rho\pm\frac{\nu}{2}, & -n+l+1\,; \\ & 2l+2\,;\end{matrix}\ \ -\frac{8z}{np^2ao}\right] \tag{5.3}$$

provided $-Re(\nu)<Re(\rho)<\frac{3}{2}$, $p>o$ and $n\geq l+1$.

Further putting $\rho=o$, we have

$$= \sum_{s=o}^{\infty} \frac{2^{2\rho}\Gamma(1-2\rho)\,\Gamma\left(\rho+\frac{\nu}{2}+\frac{\mu}{2}\right)\nu J_\nu(pt)}{p^{2\rho}\,\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\mu}{2}-\frac{\nu}{2}-\rho\right)}$$

$$\times F\left[\begin{matrix}\rho+\frac{\nu}{2}+\frac{\mu}{2},\ \rho-\frac{\nu}{2}-\frac{\mu}{2},\ \rho+\frac{\mu}{2},-\frac{\nu}{2},\rho+\frac{\nu}{2}-\frac{\mu}{2}:(c)\,;\,(c');\\ \rho,\ \rho+\frac{1}{2}:(d)\,;\,(d');\end{matrix}\ \frac{-x}{p^2},\ \frac{-y}{p^2}\right] \tag{4.2}$$

The conditions of validity for (4.1) and (4.2) are the same (with $A=B$ and $A'=B'$) as specified for equations (3.1) and (3.2) respectsvely.

(II) If $y=o$; the special case $A=A'=B=B'=O$ of (3.1) and (3.2) yields

$$t^{2\rho}{}_CF_D\left[\begin{matrix}(c)\,;\\(d)\,;\end{matrix}\ xt^2\right] = \sum_{s=o}^{\infty} \frac{2^{2\rho}\Gamma\left(\rho+\frac{\nu}{2}\right)\nu J_\nu(pt)}{p^{2\rho}\Gamma\left(1+\frac{\nu}{2}-\rho\right)}$$

$$\times {}_{C+2}F_D\left[\begin{matrix}\rho+\frac{\nu}{2},\ \rho-\frac{\nu}{2},\ (c);\\ (d)\,;\end{matrix}\ \frac{-4x}{p^2}\right] \tag{4.3}$$

and

$$t^{2\rho}J_\mu(pt)\,{}_CF_D\left[\begin{matrix}(c)\,;\\(d)\,;\end{matrix}\ xt^2\right] =$$

$$\sum_{s=o}^{\infty} \frac{2^{2\rho}\Gamma(1-2\rho)\,\Gamma\left(\rho+\frac{\nu}{2}+\frac{\mu}{2}\right)\nu J_\nu(pt)}{p^{2\rho}\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\mu}{2}-\frac{\nu}{2}-\rho\right)}$$

$$\times {}_{C+4}F_{D+2}\left[\begin{matrix}\rho+\frac{\nu}{2}+\frac{\mu}{2},\ \rho-\frac{\nu}{2}-\frac{\mu}{2},\ \rho+\frac{\mu}{2}-\frac{\nu}{2},\ \rho+\frac{\nu}{2}-\frac{\mu}{2};\ (c);\\ \rho,\ \rho+\frac{1}{2},\ (d);\end{matrix}\ \frac{x}{\rho^2}\right] \tag{4.4}$$

where both (4.3) and (4.4 are valid under the same conditions as given for equations (3.1) and (3.3) respectively with $A=A'=B=B'=C'=D'=O$.

Now using (2.1) and the orthogonal property of Bessel functions [5, p. 291,(6)], viz.,

$$\int_0^\infty x^{-1} J_{v+2m+1}(bx) J_{v+2n+1}(bx) dx \begin{cases} = O(m \neq n) \\ = \frac{1}{2v+4n+2} (m=n) \\ Re(v)+m+n<-1, \end{cases} \quad (3\cdot5)$$

weobtain

$$A_d = \frac{2^{2\rho} D \Gamma\left(\rho+\frac{D}{2}\right)}{p^{2\rho} \Gamma\left(1+\frac{D}{2}-\rho\right)} F\left[\begin{matrix} \rho+\frac{D}{2}, \rho-\frac{D}{2}, (a), (a') : (c) ; (c') ; \\ (b), (b') : (d) ; (d') ; \end{matrix} \frac{-4x}{p^2}, \frac{-4y}{p^2}\right] \quad (3\cdot6)$$

where $D = v + 2d + 1$.

On substituting the value of A_s from (3.6) in (3.3), we get (3.1).

Similarly using the intergal (2.2) and the orthogonal property of Bessel functions, we evaluate the expansion formula (3.2).

4. PARTICULAR CASES:

(I) For $a=b$ and $a'=b'$, the double hypergeometric function the L.H.S. of equations (3.1) and (3.2) breaks up into the product of two generalized hypergeometric functions and we thus get

$$t^{2\rho} {}_C F_D \left[\begin{matrix} (c) ; \\ (d) ; \end{matrix} xt^2\right] {}_{C'}F_{D'} \left[\begin{matrix} (c') ; \\ (d') ; \end{matrix} yt^2\right] = \sum_{s=0}^{\infty} \frac{2^{2\rho} \Gamma\left(\rho+\frac{v}{2}\right) v J_v(pt)}{p^{2\rho} \Gamma\left(1+\frac{v}{2}-\rho\right)}$$

$$\times F\left[\begin{matrix} \rho+\frac{v}{2}, \rho-\frac{v}{2} : (c) ; (c') ; \\ : (d) ; (d') ; \end{matrix} \frac{-4x}{p^2}, \frac{-4y}{p^2}\right] \quad (4\cdot1)$$

and

$$t^{2\rho} J_\mu(pt) {}_C F_D \left[\begin{matrix} (c) ; \\ (d) ; \end{matrix} xt^2\right] {}_{C'}F_{D'} \left[\begin{matrix} (c') ; \\ ('d) ; \end{matrix} yt^2\right]$$

and

$$t^{2\rho}J_{\mu}(pt)\,\mathbf{F}\left[\begin{array}{l}(a),(a'):(c);(c');\\ \qquad\qquad\qquad\qquad xt^2, yt^2\\ (b),(b'):(d);(d');\end{array}\right]$$

$$=\sum_{s=o}^{\infty}\frac{2^{2\rho}\Gamma(1-2\rho)\Gamma\left(\rho+\frac{v}{2}+\frac{\mu}{2}\right)vJ_v(pt)}{p^{2p}\left(1+\frac{v}{2}+\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{v}{2}-\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\mu}{2}-\frac{v}{2}-\rho\right)}$$

$$\times\mathbf{F}\left[\begin{array}{l}\rho+\frac{v}{2}+\frac{\mu}{2},\ \rho-\frac{v}{2}-\frac{\mu}{2},\ \rho+\frac{\mu}{2}-\frac{v}{2},\ \rho+\frac{v}{2}-\frac{\mu}{2}\\ \qquad\qquad\qquad\rho,\ \rho+\frac{1}{2},\end{array}\right.$$

$$\left.\begin{array}{l}(a),(a'):(c);(c');\\ \qquad\qquad\qquad\qquad \frac{-x}{p^2},\frac{-y}{p^2}\\ ,(b),(b'):(d);(d');\end{array}\right] \qquad (3\cdot2)$$

where (3.1) and (3.2) are valid for the same conditions as specified for equations (2.1) and (2.2) respectively along with $\rho\geqslant o$ and $v=v+2s+1$.

proof: To prove (3.1), let

$$f(t)=t^{2\rho}\,\mathbf{F}\left[\begin{array}{l}(a),(a'):(c);(c');\\ \qquad\qquad\qquad\qquad xt^2, yt^2\\ (b),(b'):(d);(d');\end{array}\right]=\sum_{s=o}^{\infty}A_sJ_{v+2s+1}(pt),$$

$$o<t<\infty,\ \rho\geqslant o \qquad (3\cdot3)$$

Equation (3.3) is valid, since $f(t)$ is continuous and of bounded variation in the open interval (o,∞) where $\rho\geqslant o$.

Multiplying both sides of (3.3) by $t^{-1}J_{v+2d+1}(pt)$ and integrating with respect to t between the limits o to ∞, we have

$$\int_{o}^{\infty}t^{2\rho-1}J_{v+2d+1}(pt)\mathbf{F}\left[\begin{array}{l}(a),(a'):(c);(c');\\ \qquad\qquad\qquad\qquad xt^2, yt^2\\ (b),(b'):(d);(d');\end{array}\right]dt=$$

$$=\sum_{s=o}^{\infty}A_s\int_{o}^{\infty}t^{-1}J_{v+2s+1}(pt)J_{v+2d+1}(\rho t)dt. \qquad (3\cdot4)$$

$$\int_0^\infty t^{2\rho-1}J_\mu(pt)J_\nu(pt)\,F\left[\begin{matrix}(a),(a'):(c);(c');\\ (b),(b'):(d);(d');\end{matrix}\; xt^2, yt^2\right]dt$$

$$=\frac{2^{2\rho-1}\Gamma(1-2\rho)\Gamma\left(\rho+\frac{\nu}{2}+\frac{\mu}{2}\right)}{p^{2\rho}\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}-\rho\right)\Gamma\left(1+\frac{\mu}{2}-\frac{\nu}{2}-\rho\right)}$$

$$\times F\left[\begin{matrix}\rho+\frac{\nu}{2}+\frac{\mu}{2},\ \rho-\frac{\nu}{2}-\frac{\mu}{2},\ \rho+\frac{\mu}{2}-\frac{\nu}{2},\ \rho-\frac{\mu}{2}+\frac{\nu}{2}\ (a),(a'):(c);(c')\\ \rho,\frac{1}{2}+\rho,\ (b),(b'):(d);(d');\end{matrix}\ \frac{-x}{p^2},\frac{-y}{p^2}\right] \tag{2.2}$$

where

$A+A'+C\leqslant B+B'+D$, $A+A'+C'\leqslant B+B'+D'$ $-Re(\nu+\mu)<Re(\rho)<1$ and $\rho>0$.

To establish the first integral, express the Kampé de Férier function on the L.H.S. of (2.1) in the series from and then change the order of integration and Summations which is justified [2,p. 500]. Finally, solving the inner integral with the help of formula [1, p. 326, (1)] and simplifying, we get the desired integral.

Proceeding on similar lines as above and using the result [1, p. 331, (33)] we get the integral given in Eq. (2.2).

3. THE EXPANSION FORMULAE:

The expansions to be established in this section are

$$t^{2\rho}F\left[\begin{matrix}(a),(a'):(c);(c');\\ (b),(b'):(d);(d');\end{matrix}\; xt^2, yt^2\right]=\sum_{s=0}^{\infty}\frac{2^{2\rho}\Gamma\left(\rho+\frac{\nu}{2}\right)\nu J_\nu(pt)}{p^{2\rho}\Gamma\left(1+\frac{\nu}{2}-\rho\right)}$$

$$\times F\left[\begin{matrix}\rho+\frac{\nu}{2},\rho-\frac{\nu}{2},(a),(a'):(c);(c');\\ (b),(b'):(d);(d');\end{matrix}\ \frac{-4x}{p^2},\frac{-4y}{p^2}\right] \tag{3.1}$$

and adopt the following contracted notations of Burchnall and Chaundy [3] for Kampé de Fériet function:

$$F\left[\begin{matrix}(a),(a'):(c);(c');\\ (b),(b'):(d);(d');\end{matrix} x,y\right] =$$

$$= \sum_{m=o}^{\infty}\sum_{n=o}^{\infty} \frac{((a))_{m+n}((a'))_{m+n}((c))_m((c'))_n x^m y^n}{((b))_{m+n}((b'))_{m+n}((d))_m((d'))_n (m)!\,(n)!},$$

where $A+A'+C \leqslant B+B'+D$, $A+A'+C' \leqslant B+B'+D'$ and (a) denotes the sequence of A parameters

$$a_1, \ldots\ldots\ldots, a_A$$

Hence there are A of a parameters, A' of a' parameters and so on. Further $((a))_m$ is to be interpreted as $\prod_{j=1}^{A}(a_j)_m$, with similar interpretations for $((a'))_m$ etc.

In Section 2, We derive two integrals which shall be used in section 3 in evaluating the desired expansion formulae. In section 5, we apply one of the paraticular cases obtained in section 4 in radial wave function for Hydrogen like atoms.

2. DERIVATION OF THE INTEGRALS:

In this section, we obtain the following integrals to be used later in evaluating the expansion formulae

$$\int_0^{\infty} t^{2\rho-1} J_\nu(pt)\, F\left[\begin{matrix}(a),(a'):(c);(c');\\ (b),(b'):(d);(d').\end{matrix} xt^2, yt^2\right] dt$$

$$= \frac{2^{2\rho-1}\Gamma\left(\rho+\frac{\nu}{2}\right)}{p^{2\rho}\Gamma\left(1+\frac{\nu}{2}-\rho\right)} F\left[\begin{matrix}\rho+\frac{\nu}{2}, \rho-\frac{\nu}{2}, (a),(a'):(c);(e');\\ (b),(b'):(a);(a');\end{matrix} -\frac{4x}{p^2}, -\frac{4y}{p^2}\right] \quad (2\cdot1')$$

provided

$$A+A'+C \leqslant B+B'+D,\ A+A'+C' \leqslant B+B'+D', -Re(\nu) < Re(\rho) < \frac{3}{2}$$

and $\rho > 0$.

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ON EXPANSION FORMULAE FOR GENERALIZED HYPERGEOMETRIC AND RADIAL WAVE FUNCTIONS

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ABSTRACT: Two expansion formulae for Kampé de Feriét function have been evaluated with the help of orthogonal property of Bessel functions. A few interesting cases along with the application of one of them in the radial wave function for the hydrogen like atoms have also been discussed.

1. INTRODUCTION:

The present paper is inspired by the frequent requirement of various properties of special functions which play a vital role in the study of potential and other allied problems in Quantum mechanics. The Appell's function and the functions related to them have many applications in Mathematical physics [4], [7], and [8] and the authors evaluate here two summation formulae for Kampé de Feriét function by making use of the orthogonal property of Bessel functions.

A few particular cases of interest from the point of view of their applicability in Quantum mechanics have also been discussed in section 4.

Similar work on Kampé de Feriét function in evaluating certain expansion formula in terms of Jacobi polynomials and generalized hypergeometric function has been recently done by Singh and Sharma [11].

In what follows we make use of the familiar abbreviations

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1)$$